

Instructions:

Please write your answers on separate paper. Please write clearly and legibly, using a large font and plenty of white space (I need room to put my comments). Staple all your pages together, with your problems in order, when you turn in your exam. Make clear what work goes with which problem. Put your name or initials on every page. To get credit, you must show adequate work to justify your answers. If unsure, show the work. Simplify your answers, except you may leave any number of more than three digits (like 123×456 or 15^4) unsimplified. No outside materials are permitted on this exam – no notes, papers, books, calculators, phones, smartwatches, or computers – only pens and pencils, and your coursepack. You may use any result in the coursepack (whether boxed or an exercise). However, you must cite it, and you may not use it to prove itself (or a portion/special case of itself). Each problem is out of 10 points, 100 points maximum. You have 75 minutes.

1. A hand in the card game bridge consists of a set of 13 cards, from the standard 52-card deck. What is the probability that a hand contains none of $\{10, J, Q, K, A\}$?
2. We consider words from the five-letter alphabet $\{A, B, C, D, E\}$. How many seven-letter words are there that contain exactly one double letter?
3. We consider words from the five-letter alphabet $\{A, B, C, D, E\}$. How many seven-letter words are there that do NOT use all five letters from the alphabet?
4. Consider the Mersenne numbers $M_i = 2^i - 1$, for $i \geq 1$. Prove that some Mersenne number is divisible by 75.
5. Seven points are chosen from a regular hexagon with side length 1cm. Prove that some pair of these points must be within distance 1cm of each other.
6. How many solutions are there to the equation $x_1 + x_2 + x_3 = 31$, where x_1 is a nonnegative even integer, x_2 is a positive even integer, and x_3 is a positive odd integer?
7. Find, with proof, a simple formula for $\binom{a}{2}$ ($a \geq 2$).
8. Let $b \in \mathbb{Z}$. Prove that $\binom{a}{b} = (-1)^b \binom{b-a-1}{b}$.
9. Calculate, with justification, $\binom{7}{4}$. You may not use the table in the back of your coursepack, but you may use any formulas proved in the exercises.
10. Find each partition of 10 into exactly four parts, identify any self-conjugate ones, and draw the Ferrers diagrams for those self-conjugate ones.